ON RETARDED GREEN'S FUNCTION FOR COVARIANT KLEIN-GORDON EQUATION

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It is shown that, in case of special classes of static metrics, the fundamental solution (Green's function) for the Klein-Gordon equation can be derived by means of the massless fundamental solution. In applying the obtained result to the weak gravitational field the corresponding Green's function is computed and some of its global properties are analyzed.

1.Introduction

The Green's function of the covariant wave equation is of great importance in mathematical and physical applications. For example, the propagation of non-gravitational fields (the Maxwell field, the massive scalar field, etc.) in a gravitational background can be investigated by means of one [1]. Such situation may be found in the vicinity of superdense astrophysical objects where the curvature of space-time is not negligible [2]. An important problem of the wave propagation - the validity of Huygens` principle - can be analyzed with the aid of Green's functions, too [3]. Further, if the radius of curvature is so small that one is comparable with Compton length of the non-gravitational field, we must develop the quantisation of corresponding field on a curved space-time. Now the covariant Green's functions of the equations for the classical field are due to build up the different quantum theory quantities [4,5].

The present paper deals with the Green's functions (G) for the massive scalar field, which satisfy the covariant Klein-Gordon (KG) equation

$$\hat{L}G(x,y) := (\Box + \xi R(x) + m^2)G(x,y) = \delta^4(x,y) \qquad \textbf{(1)}$$

with m the mass of the field and ξ a constant. Here d'Alembertian $\Box = g^{ik}(x)\nabla_i\nabla_k$, where $g^{ik}(x)$ is a metric tensor of a pseudo-Riemannian space V^4 with a signature (+,-,-,-) and ∇_i denotes the covariant derivative. The scalar curvature has the opposite sign in comparison with the scalar curvature in [5,6] and $\delta^4(x,y)$ is Dirac delta distribution in V^4. All differentiations of two-point functions refer to the first argument and small Latin indices run from 0 to 3.

The theory of covariant Green's functions is proposed in the works [1,3,6]. According to the mentioned works, on a causal domain $\Omega \subseteq V^4$ (see [6]), the retarded and advanced Green's functions (fundamental solutions) are given by expression

$$G^{\pm}(x,y) = \frac{1}{4\pi}\left(W(x,y)\,\delta^{\pm}(\sigma(x,y)) + V^{\pm}(x,y)\right) .$$

The world function $\sigma(x,y)$ is equal to half square root of the geodesic distance between the points x,y. σ is negative for space-like intervals, positive for time-like ones and satisfies the equation $\nabla^i\sigma\nabla_i\sigma = 2\sigma$. Transport scalar W(x,y) coincides with the scalarised Van Vleck determinant and satisfies the transport equation with the additional condition [6]:

$$\begin{aligned} (2\nabla^i\sigma\,\nabla_i + (\Box\sigma - 4))\,W(x,y) = 0\ , \ \ \forall x,y\in\Omega, \\ W(y,y) = 1\ \ . \end{aligned} \qquad \textbf{(2)}$$

Delta distributions $\delta^{\pm}(\sigma(x,y))$ have supports on the future light-cone $C^+(y)$ and on the past light-cone $C^-(y)$ respectively. The tail terms $V^{\pm}(x,y)$ have supports in closures $J^{\pm}(y)= C^{\pm}(y) \cap D^{\pm}(y)$, where $D^{\pm}(y)$ denotes interiors of the cones $C^{\pm}(y)$, and are determined by the characteristic Cauchy problem [6]. In the region $D^{\pm}(y)$ the functions $V^{\pm}$ must satisfy homogeneous differential equation

$$\hat{L}\,V^{\pm}(x,y) = 0\ \ , \qquad \textbf{(3)}$$

which is completed by characteristic initial conditions

$$\begin{aligned} \hat{P}V^{\pm}(x,y) := (2\nabla^i\sigma\nabla_i + (\Box\sigma - 2))V^{\pm}(x,y) = -\hat{L}W(x,y), \\ \forall\ x\in C^{\pm}(y). \end{aligned} \qquad \textbf{(4)}$$

We shall only discuss V^+ because the corresponding results for V^- can be deduced by reversing the time orientation of Ω and hence we can omit the notation "±".

As the explicit calculation of *G* is quite difficult it has been done only for particular metrics [5,6,8]. To get an information about the fundamental solutions in less special space-times different approximation methods are valuable.

In Hadamard's method $V(x,y)$ is expanded up as a power series in σ [6,7]. That technique has been used by DeWitt [1,4], Günther [3], John [9] and others for different wave equations. Such approach is true near the light-cone where $\sigma \to 0$. For example, in discussing regularization procedures in quantum field theory the Green's functions of near arguments $(x \to y)$ are used and above-mentioned way is sufficient [4,5]. In other applications (particle production, interacting fields), where the information about nonlocal effects is required, one would use a different method.

An alternative expansion procedure is based on the assumption that we have a small perturbation on the Minkowskian metric. Such approach was used for the massless scalar field and vector field in the works [3,10 - 14]. In the present paper we shall apply the perturbation method to the Klein-Gordon equation.

2. The first order approximation

We assume that the gravitational field is weak and the metric tensor can be expanded up in the small parameter ε:

$$g_{ik}(x) = \underset{0}{g_{ik}}(x) + \varepsilon \gamma_{ik}(x) + 0(\varepsilon^2) \ .$$

The parameter ε marks the order of deviation from the metric tensor of flat space-time $\underset{0}{g_{ik}}(x)$. Here and in the following, index 0 denotes the quantities of a flat space-time. All quantities which depend on metric will be expanded up to the first order in ε, too. It must be pointed out that $\underset{0}{W} \equiv 1$ and the functions $\underset{0}{\sigma}$, $\underset{1}{\sigma}$, $\underset{1}{W}$ are given in the works [11,12].

Now we shall use the approximation method to find the first order tail term for the Klein-Gordon equation. Let us search for V of the form

$$V(x,y) = W(x,y)\, Z(\sigma(x,y)) + \varepsilon \underset{1}{V}(x,y) + 0(\varepsilon^2) \ ,$$

where

$$Z(\sigma) = -\frac{m}{\sqrt{2\sigma}} J_1(m\sqrt{2\sigma}) \qquad \textbf{(5)}$$

and J_1 denotes the Bessel function of first order. The function (5) is similar to the tail term on a Minkowskian space-time, $\underset{0}{Z}(\sigma) = \underset{0}{V}(x,y)$.

After expanding up the equations (3,4), the characteristic initial value problem for $\underset{1}{V}$ becomes:

$$\varepsilon \underset{0}{\hat{L}}\, \underset{1}{V}(x,y) = -\underset{0}{V}(x,y)\, \hat{L}_M W(x,y) \ , \qquad x \in D^{+}(y) \ ,$$

$$\varepsilon \underset{0}{\hat{P}}\, \underset{1}{V}(x,y) = -\hat{L}_M W(x,y) \ , \qquad x \in C^{+}(y) \ ,$$

where $\hat{L}_M$ denotes the wave operator for the massless field, $L_M := \hat{L} - m^2$. As on the Minkowskian space-time the tail term of the fundamental solution for the KG equation is known, it is just $\underset{0}{v}$, we can solve the characteristic Cauchy problem for $\underset{1}{v}$. Let us use the integral representation of the solution of the non-homogeneous characteristic initial value problem, given in the book [6]. To simplify the notations, we shall set

$$\Omega_0 := J^{-}(x) \cap J^{+}(y) \ , \qquad \Sigma_y := J^{-}(x) \cap C^{+}(y) \ ,$$
$$\Sigma_x := C^{-}(x) \cap J^{+}(y) \ , \qquad S := C^{-}(x) \cap C^{+}(y) \ .$$

Note that $\partial\Omega_0 = \Sigma_x \cup \Sigma_y$ and the 2-surface $S = \partial\Sigma_x = \partial\Sigma_y$. Now, according to [6], $\underset{1}{V}$ is the sum of two terms:

$$\underset{1}{V} = \underset{1}{V}' + \underset{1}{V}'' \ .$$

The first summand is derived from the initial condition:

$$\varepsilon \underset{1}{V}' = -\frac{1}{4\pi}\int_S \hat{L}_M W \omega(z) - \frac{1}{4\pi}\int_{\Sigma_y} \underset{0}{V}(x,z)\hat{L}_M W \mu_y(z) \ . \tag{6}$$

The other one is

$$\varepsilon \underset{1}{V}'' = -\frac{1}{4\pi}\int_{\Sigma_x} \underset{0}{V}(z,y)\hat{L}_M W \mu_x(z) - \frac{1}{4\pi}\int_{\Omega_0} \underset{0}{V}(x,z)\underset{0}{V}(z,y)\hat{L}_M W \mu(z). \tag{7}$$

Here $W \equiv W(z,y)$, a Leray form μ_x and 2-form $\omega(z)$ are defined by

$$d\sigma(z,x) \wedge \mu_x(z) = d\sigma(z,y) \wedge d\sigma(z,x) \wedge \omega(z) = \mu(z)$$

and μ(z) is an invariant volume element. It clearly follows from (5-7) that $\underset{1}{V}^{//}$ and the second integral in $\underset{1}{V}^{/}$ will vanish when m=0. This implies that the first integral in (6) is the tail term for the massless field. It also coincides with the integral, calculated in the work [12].

However, in general case the integrals in (6,7) are cumbersome for the physical applications. In the next sections we shall consider a static metric, when the tail term of G will take more simple form and some physical conclusions will be possible.

3. Fundamental solution in case of the static metric

In this section we shall discuss the fundamental solution in case of the special classes of static metric. We suppose that there exists a coordinate system, where the metric in question can be given by the line element as

$$ds^2 = dx^{0\,2} + g_{\alpha\beta} dx^{\alpha} dx^{\beta} \ . \qquad \textbf{(8)}$$

Greek indices run from 1 to 3 and $g_{\alpha\beta}$ depends on the space coordinates x^{α}, only.

Usually the cosmological constant Λ is taken 0 and then such kind of metric does not describe real gravitational field, which must satisfy the Einstein field equations. But the method developed here should also be applicable to problems outside the gravitational physics (e.g., wave diffusion in non-homogeneous media). When one takes $\Lambda \neq 0$, the cosmological applications of (8) are possible.

We shall use the property of the metric (8), that the corresponding regular term of the fundamental solution for the KG equation is simply found by means of the regular term of the fundamental solution for massless wave equation (V_M). It simplifies the analyze of G because V_M has more elementary form, usually. For example, when V_M is known for the metric, conformal to the metric (8), that attribute can be used in case of the conformally invariant wave equation (1) (ξ=-1/6). When the metric depends on a small parameter and the perturbation technique is possible one can apply the mentioned relation, too.

Theorem. Let the metric of a pseudo-Riemannian space V^4 be given by (8), and let x,y be the points in a causal domain $\Omega \subset V^4$. Then the regular part of the fundamental solution for the Klein-Gordon equation can be represented as

$$V(x,y) = W\,Z(\sigma) + V_M(x,y) + \int_0^{\sigma} [\![V_M(x,y)]\!]\,Z(\sigma - s)\,ds \ , \qquad \textbf{(9)}$$

where $[\![V(x,y)]\!] \equiv V(x,y)_{|\,\sigma = s}$.

The proof of the theorem is quite elementary if one notes that in case of the metric (8) $\Box\,\sigma$ and W do not depend on the time-coordinate x^0. Therefore they are independent of the choice of the hypersurface $\sigma = s$ = const ≥ 0 and we can write

$$[L_M^{\Box} W]_{|\sigma = 0} = \hat{L}_M W \ .$$

Using the transport equation (2) and the fact that V_M is the solution of the respective characteristic Cauchy problem,

$$\hat{L}_M V_M = 0 \ , \ \forall \ x \in D^{+}(y) \ ; \qquad \hat{P}\,V_M = -\hat{L}_M W \ , \ \forall \ x \in C^{+}(y) \ ,$$

it is possible to show that the right term in the expression (9) satisfies the differential equation (3) with the initial conditions (4). Therefore, by virtue of the uniqueness of the solution of the characteristic initial problem [6], (9) is the regular term of G for the KG equation. We should note that it is useful to realize the calculations in the coordinates $x'^0 = \sigma$, $x'^\alpha = x^\alpha$.

To illustrate the potentialities of the solution (9), let us concern the Einstein universe. Taking into account that R=const, we can write the equation (1) as

$$(\Box - \frac{1}{6} R + m^{*2})\, G(x,y) = \delta^4(x,y) \ ,$$

where we have used the notation $m^{*2} = m^2 + (\xi + \frac{1}{6}) R$. As the Einstein universe is conformally flat, V_M=0, and (9) gives us the known formula $V = W Z(\sigma)$ with Z defined by (5), but m replaced by m^* (see, e.g.,[8]). The second demonstration about applications of (9) is the conformal perturbation of the background metric (8), when the line element can be given of the form

$$ds^2 = (1 + \varepsilon \Psi(x))(dx^{02} + g_{\alpha\beta}\, dx^{\alpha} dx^{\beta})$$

and ε is a small parameter. As an example we shall turn to the weak static gravitational field in the next section.

4. Fundamental solution in a weak static gravitational field

Let us discuss a weak static gravitational field with the metric

$$ds^2 = (1 + 2\Psi(\vec{r}))[dx^{0\,2} - (1 - 4\Psi(\vec{r}))\,d\vec{r}^{\,2}] \ . \qquad \textbf{(10)}$$

Here $d\vec{r}^{\,2}$ is the metric of the Euclidean 3-space and $\Psi(\vec{r})$ is the Newtonian potential, satisfying the equation

$$\Delta\Psi(\vec{r}) = 4\pi\varkappa\rho(\vec{r}) \ .$$

$\varkappa$ is the gravitational constant, ρ - mass density and Δ - the Laplacian belonging to the Euclidean 3-space. Here and in the following, linear terms in $\varkappa$ will be taken into account, only.
It is seen that (10) is conformal to the line element

$$d\tilde{s}^2 = dx^{0\,2} - (1 - 4\Psi(\vec{r}))\,d\vec{r}^{\,2} \ , \qquad \textbf{(11)}$$

which is evidently particular case of (8). Now we can do the conformal transformation in the equation (1):

$$(\tilde{\Box} - \frac{1}{6}\tilde{R} + \tilde{C} + m^2)\,\tilde{G}(x,y) = \tilde{\delta}^4(x,y) \ .$$

Here "~" denotes the quantities which refer to the metric (11) and $\tilde{C} \equiv (\xi + \frac{1}{6})R + 2m^2\Psi$.

According to the theorem, proposed in the previous section, one can write

$$\tilde{V}(x,y) = \tilde{W}\,Z(\tilde{\sigma}) + \tilde{V}_M + \int_0^{\tilde{\sigma}} [\![\, \tilde{V}_M \,]\!]\, Z(\tilde{\sigma} - s)\,ds \ ,$$

where $\tilde{V}_M$ is the solution of the respective Cauchy problem:

$$\tilde{L}_M \tilde{V}_M \equiv (\tilde{\Box} - \frac{1}{6}\tilde{R} + \tilde{C})\,\tilde{V}_M = 0 \ , \qquad x \in D^+(y) \ ,$$

$$\tilde{\hat{P}}\,\tilde{V}_M = -\tilde{\hat{L}}_M\,\tilde{W} \ , \qquad x \in C^+(y) \ .$$

Note that $\tilde{R}$ and $\tilde{C}$ are proportional to ϰ, thus they are small of the first order. It makes possible to find $\tilde{V}_M$ by the approximation method, proposed in the work [12] (see the section 2 of this paper, too).

It is known that under a conformal transformation of the metric there is a simple relation between the corresponding fundamental solutions [6]:

$$G(x,y) = (1+2\Psi(x))^{-\frac{1}{2}}(1+2\Psi(y))^{-\frac{1}{2}}\tilde{G}(x,y). \tag{12}$$

Using the transformation direction (12) and relations between the quantities $\tilde{\sigma}$, $\tilde{W}$ and σ, W, the regular part V can be expressed, in the first order in ϰ, as the sum of two terms:

$$V(x,y) = W\,Z(\sigma) + V^{*}(x,y)\ , \tag{13}$$

where the first is similar to the tail term on Minkowskian space-time. The other one can be written as

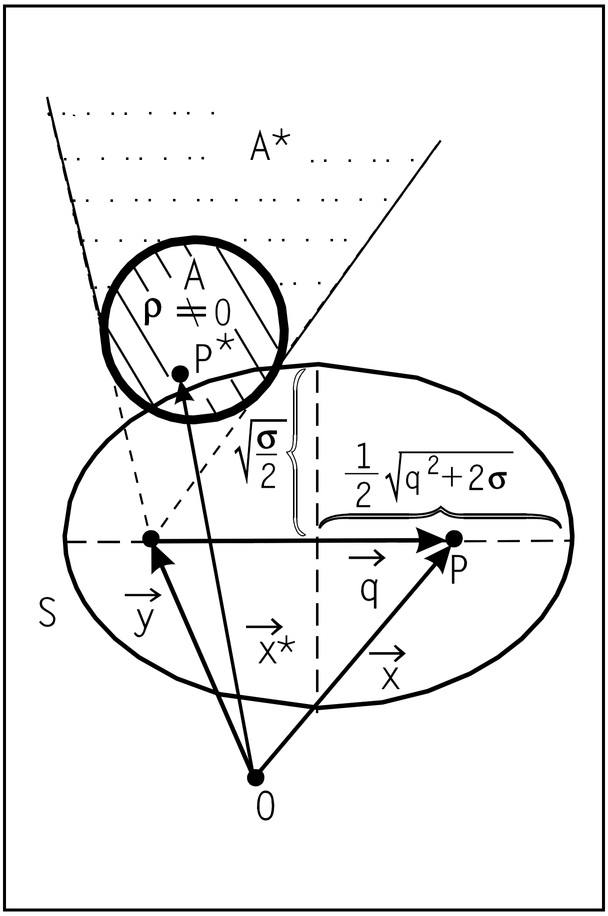
$$\begin{aligned} V^{*}(x,y) \equiv &-\frac{\xi}{4\pi}\left\{\int_S R\,\omega + \int_0^{\sigma} [\![\textstyle\int_S R\omega]\!]\, Z(\sigma-s)\,ds\right\} + \\ &+2\varkappa\left(\frac{1}{\sigma}M(\sigma,\vec{q})\right)_{,x^0} + \\ &+2\varkappa\int_0^{\sigma}\left(\frac{\sigma+q^2+s}{s\sqrt{q^2+2s}}M(s,\vec{q})\right)_{,s} Z(\sigma-s)\,ds\ . \end{aligned} \tag{14}$$

Here we use the notation $\vec{q} \equiv \vec{x}-\vec{y}$ and comma denotes the partial derivative. The calculations to obtain the expressions (13,14) are rather lengthy but not very complicated.

The quantity

$$M(\sigma,\vec{q}) \equiv \int_{\Sigma_y} \rho(\vec{x}^{\,*})\,d\Sigma_0(\vec{x}^{\,*}) \tag{15}$$

is just that portion of the gravitating mass which stays inside the ellipsoid $\Sigma_y \equiv C^+(y) \cap J^-(x)$ (see the figure). $d\Sigma_0$ denotes the time-like component of the invariant surface element on Σ_y. Note that the 2-surface $S \equiv C^+(y) \cap C^-(x)$ is the boundary of Σ_y.



It should be pointed out that the first term on the right-hand side of (14) vanishes in case of minimal coupling of the gravitational and scalar fields (ξ=0). The second summand in (14) coincides with the regular part of the fundamental solution for massless scalar field [14] and the last integral describes the special action of the gravitational field on the massive field.

It follows from (14,15) that V^* has some properties similar to those of V_M. For example, when the source of the gravitational field is localized in a world-tube Γ (the region A on the figure) and the space-time region $D \equiv \{x\colon x \in J^+(y), \Sigma_y \cap \Gamma = \varnothing\}$, then $V^*(x,y)$ vanishes for every $x \in D$.

5. Point-mass approximation

The point-mass approximation for gravitational sources is applicable, if the space-like distances of the points x and y from the world-tube Γ are much greater that the space-like dimensions of Γ.

Let us now assume that the gravitational field is produced by a source A, concentrated at the point of the 3-space $P^* = (\vec{x}^{\,*})$. In that case the condition $0 \le q^0 < |\vec{q} - \vec{r}_0| + r_0$ corresponds to the region D, defined in the previous section. Here $q^0 \equiv x^0 - y^0$ and $\vec{r}_0 \equiv \vec{x}^{\,*} - \vec{y}$. Now it can be seen that V^* does not vanish at the point $x = (x^0, \vec{x})$ for the time coordinates $x^0 \ge y^0 + r_0 + |\vec{q} - \vec{r}_0|$, only. Hence the tail term V^* delayed behind the singular impulse $W\delta(\sigma)$ by a time

$\Delta\tau = |\vec{q} - \vec{r}_0| + r_0 - q$. The analogous result for the massless scalar field is obtained in the work [12] and for the electromagnetic field in the work [13].

At the instants $q^0 > |\vec{q} - \vec{r}_0| + r_0$ we shall be limited to the minimal coupling of the gravitational and scalar fields (ξ=0). It is evident that V^* takes the simple form when the point $P = (\vec{x})$ lies inside the 3-space region $\sigma_1 \equiv \frac{1}{2}[(|\vec{q} - \vec{r}_0| + r_0)^2 - q^2] > 0$. Note that the "line" $\sigma_1 = 0$ corresponds to the "shadow" A^* of the source A on the figure, brought in the previous section. If $P \in A^*$, the point-mass approximation cannot be applied to (14).

When $q^0 > |\vec{q} - \vec{r}_0| + r_0$ and $\sigma_1 > 0$ the formulae (14) and (15) give:

$$V^*(x,y) = -\varkappa M q^0 \sigma^{-2}(2 + m^2\sigma) +$$
$$+\varkappa M m^2 \int_{\sigma_1}^{\sigma} \frac{\sigma + q^2 + s}{s(\sigma - s)\sqrt{q^2 + 2s}} J_2(m\sqrt{2(\sigma - s)})\, ds \;.$$

This leads to an interesting conclusion in ξ=0 case: inside the space-time region $q^0 > |\vec{q} - \vec{r}_0| + r_0, \sigma_1 > 0$ the structure of V^* depends on q^0, $\vec{q}$ and total mass *M*, only. The internal structure of the source A is of no consequence for this case.

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KOVARIANTSE KLEIN-GORDONI VÕRRANDI RETARDEERITUD GREENI FUNKTSIOON

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Teatud klassi staatiliste meetrikate korral on tuletatud lihtne seos kovariantse Klein-Gordoni vôrrandi ja vastava ilma massita välja vôrrandi fundamentaallahendite (Greeni funktsioonide) vahel. Toetudes saadud tulemustele, on leitud Klein-Gordoni vôrrandi retardeeritud Greeni funktsioon nôrga gravitatsioonivälja lähenduses ja analüüsitud selle môningaid globaalseid omadusi.

РЕТАРДИРОВАННАЯ ФУНКЦИЯ ГРИНА КОВАРИАНТНОГО УРАВНЕНИЯ КЛЕЙНА-ГОРДОНА

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Показано, что в случае определенного класса статических метрик существует простая связь между фундаментальным решением (функции Грина) ковариантного уравнения Клейна-Гордона и фундаментальным решением волнового уравнения для безмассового поля. На основе найденного результата получено выражение для

ретардированной функции Грина в приближении слабого гравитационного поля и анализированы его некоторые глобальные свойства.